6.1 Areas Between Curves

Area between the graphs:

$$\int_a^b (f(x) - g(x))\, dx, \quad f(x) \geq g(x) \text{ on } [a, b]$$

Integral along the y-axis:

$$\int_c^d (g(y) - h(y))\, dy, \quad g(x) \geq h(x) \text{ on } [c, d]$$

6.2 Volumes

Volume:

$$\int_a^b A(y)\, dy, \quad A(y) \text{ is the cross-sectional area}$$

Mass:

$$\int_a^b \rho(x)\, dx, \quad \rho(x) \text{ is the linear mass density}$$

Population:

$$2\pi \int_0^R r\rho(r)\, dr, \quad \rho(r) \text{ is the radial density}$$

Flow Rate:

$$2\pi \int_0^R rv(r)\, dr, \quad v(r) \text{ is the velocity at radius } r$$

Average Value:

$$\frac{1}{b-a} \int_a^b f(x)\, dx, \quad f(x) \text{ is continuous}$$

6.3 Volumes of Revolution

Disk Method:

$$\pi \int_a^b (f(x)^2 - g(x)^2)\, dx, \quad \text{Outer radius } f(x) \text{ and inner radius } g(x)$$

Rotation about $y = c$:

$$\pi \int_a^b ([c - f(x)]^2 - [c - g(x)]^2)\, dx, \quad c \geq f(x) \geq g(x)$$

$$\pi \int_a^b ([f(x) - c]^2 - [g(x) - c]^2)\, dx, \quad f(x) \geq g(x) \geq c$$

6.4 Cylindrical Shells

Shell Method:

$$2\pi \int_a^b x[f(x) - g(x)]\, dx, \quad f(x) \geq g(x) \geq 0$$

Rotation about $x = c$:

$$2\pi \int_a^b (x - c)[f(x) - g(x)]\, dx, \quad c \leq a$$

$$2\pi \int_a^b (c - x)[f(x) - g(x)]\, dx, \quad c \geq a$$

7.1 Integration by Parts

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int u'(x)v(x)\, dx$$

Choose u so that u' is simpler than u.

dv must contain the 'dx'

7.2 Trigonometric Integrals

$\cos x$, $\sec x$ are even. Everything else is odd.

$\sin^2 x + \cos^2 x = 1$ $\quad$ $\tan^2 x + 1 = \csc^2 x$

$1 + \cot^2 x = \csc^2 x$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$
$$\frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x$$
$$\int \tan x\,dx = \ln|\sec x| \qquad \int \csc x\,dx = \ln|\csc x - \cot x|$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| \qquad \int \cot x\,dx = -\ln|\csc x| = \ln|\sin x|$$

$$\frac{d}{dx}\arcsin x\,dx = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x\,dx = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x\,dx = \frac{1}{x^2+1} \qquad \frac{d}{dx}\operatorname{arccot} x\,dx = -\frac{1}{x^2+1}$$
$$\frac{d}{dx}\operatorname{arcsec} x\,dx = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc} x\,dx = -\frac{1}{|x|\sqrt{x^2-1}}$$

7.3 Trigonometric Substitution

$\sqrt{x^2+c^2}$	$x = c\tan\theta$	$\sqrt{x^2-c^2}$	$x = c\sec\theta$
$\sqrt{c^2-x^2}$	$x = c\sin\theta$	$\sqrt{x^2+bx+c}$	Complete the square

7.4 Hyperbolic Integrands

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\int \sinh x\,dx = \cosh x \qquad \int \cosh x\,dx = \sinh x$$
$$\int \operatorname{sech}^2 x\,dx = \tanh x \qquad \int \operatorname{csch}^2 x\,dx = -\coth x$$
$$\int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x \qquad \int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arcsinh} x$$
$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x \qquad x > 1$$
$$\int \frac{dx}{1-x^2} = \operatorname{arctanh} x \qquad |x| < 1$$
$$\int \frac{dx}{1-x^2} = \operatorname{arccoth} x \qquad |x| > 1$$
$$\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{arcsech} x \qquad 0 < x < 1$$
$$\int \frac{dx}{|x|\sqrt{x^2+1}} = -\operatorname{arccsch} x \qquad x \neq 1$$

7.5 Partial Fractions

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

Where $\deg(P) < \deg(Q)$ and $Q(x)$ can be factored explicitly as a product of linear and irreducible quadratic terms.

If factors are not unique:

$(x-a)^n$ contributes $\dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_n}{(x-a)^n}$

$(x^2+b)^n$ contributes $\dfrac{A_1x+B_1}{x^2+b}+\dfrac{A_2x+B_2}{(x^2+b)^2}+\cdots+\dfrac{A_nx+B_n}{(x^2+b)^n}$

If $\deg(P)=\deg(Q)$, add terms from the denominator to the numerator and then subtract them again, forming a 1. If $\deg(P)>\deg(Q)$, use long division.

7.8 Numerical Integration

$y_j = f(a+j\Delta x)$

$T_N = \dfrac{b-a}{2N}(y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_N)$

$M_N = \dfrac{b-a}{N}(f(c_1)+f(c_2)+\cdots+f(c_N)) \quad c_j = a+\dfrac{(b-a)[2j-1]}{2N}$

$S_N = \dfrac{b-a}{3N}[y_0+4y_1+2y_2+\cdots+2y_{N-2}+4y_{N-1}+y_N] = \dfrac{T_{N/2}}{3}+\dfrac{2M_{N/2}}{3}$

$\text{Error}(T_N) = \dfrac{K_2(b-a)^3}{12N^2}, \quad \text{Error}(M_N) = \dfrac{K_2(b-a)^3}{24N^2}$

$\text{Error}(S_N) = \dfrac{K_4(b-a)^5}{180N^4} \quad K_N \geq \left|f^{(N)}(x)\right|$

10.1 Sequences

For sequence a_n, if $\lim_{n\to\infty} a_n$ does not exist, then a_n diverges.

A sequence converges if it is either monotonic increasing and bounded above or monotonic decreasing and bounded below.

A sequence of the form cr^n is called a *geometric sequence*.

$a_n = f(n), \quad \lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L$

The Basic Limit Laws and Squeeze Theorem can be applied to sequences.

Basic Limit Laws:

Assuming $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist,

$\lim_{x\to c} k\,f(x) = k\lim_{x\to c} f(x)$ For constant k

$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x)+\lim_{x\to c} g(x)$

$\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x)$

$\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

$\lim_{x\to c}(f(x))^{p/q} = \left(\lim_{x\to c} f(x)\right)^{p/q}, \quad p,q\in\mathbb{Z}, \quad q\neq 0$

Squeeze Theorem:

$f(x)\leq g(x)\leq h(x), \quad \lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L \implies \lim_{x\to c} g(x) = L$

L'Hôpital's Rule:

If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ or $\pm\infty$ and $\dfrac{f'(x)}{g'(x)}$ exists then

$\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = \lim_{x\to\infty}\dfrac{f'(x)}{g'(x)}$

10.2 Infinite Series

$\sum_{n=k}^{\infty} a_n$ is convergent, and converges to L if $\lim_{N\to\infty}\sum_{n=k}^{N} a_n = L$ exists.

It is divergent if the limit does not exist.

Divergence Test:

$\sum_{n=k}^{\infty} a_n$ is divergent if $\lim_{n\to\infty} a_n \neq 0$

Convergence of Geometric Series:

$\sum_{n=k}^{\infty} cr^n = \dfrac{cr^k}{1-r}, \quad |r|<1$

Geometric series diverge if $|r|\geq 1$

7.6 Improper Integrals

Convergence of Improper Integrals:

$\int_a^\infty f(x)\,dx$ converges if $\lim_{R\to\infty}\int_a^R f(x)\,dx$ exists and diverges otherwise.

If $f(x)$ is continuous on $[a,b)$ then $\int_a^b f(x)\,dx = \lim_{R\to b-}\int_a^R f(x)\,dx$

P-Integrals:

$\int_a^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges otherwise.

Comparison Test:

$f(x) \geq g(x) \geq 0$ for $x \geq a$

$\int_a^b f(x)\,dx$ converges $\Longrightarrow \int_a^b g(x)\,dx$ converges

$\int_a^b g(x)\,dx$ diverges $\Longrightarrow \int_a^b f(x)\,dx$ diverges

10.3 Convergence of Series with Positive Terms

The partial sums of a positive series form an increasing sequence.

Dichotomy Theorem:

Positive series converge $\iff$ the series is bounded.

Integral Test:

$a_n = f(n)$

f is positive, decreasing, and continuous for $x > k$.

$\int_k^\infty f(x)\,dx$ converges $\Longrightarrow \sum_{n=k}^\infty a_n$ converges.

$\int_k^\infty f(x)\,dx$ diverges $\Longrightarrow \sum_{n=k}^\infty a_n$ diverges.

P-Series:

$\sum_{n=k}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

Comparison Test:

$0 \leq a_n \leq b_n \,\forall\, n \geq k$

$\sum_{n=k}^\infty b_n$ converges $\Longrightarrow \sum_{n=k}^\infty a_n$ converges.

$\sum_{n=k}^\infty a_n$ diverges $\Longrightarrow \sum_{n=k}^\infty b_n$ diverges.

Limit Comparison Test:

$L = \lim_{n\to\infty}\frac{a_n}{b_n}$

$L > 0, \quad \sum_{n=k}^\infty a_n$ converges $\iff \sum_{n=k}^\infty b_n$ converges

$L = \infty, \quad \sum_{n=k}^\infty a_n$ converges $\Longrightarrow \sum_{n=k}^\infty b_n$ converges

$L = 0, \quad \sum_{n=k}^\infty b_n$ converges $\Longrightarrow \sum_{n=k}^\infty a_n$ converges

10.4 Absolute and Conditional Convergence

Absolute Convergence:

$\sum_{n=k}^\infty |a_n|$ converges $\Longrightarrow \sum_{n=k}^\infty a_n$ converges absolutely.
Absolute convergence $\Longrightarrow$ convergence.

Conditional Convergence:

$\sum_{n=k}^\infty a_n$ converges, $\sum_{n=k}^\infty |a_n|$ diverges $\Longrightarrow \sum_{n=k}^\infty a_n$ converges conditionally.

Alternating Series Test:

If a_n is alternating and decreasing, and $\lim_{n\to\infty} a_n = 0$, then a_n converges.

Decreasing of a series can be found by proving that $a_n > a_{n+1}$, and or through the use of the derivative of $a_n = f(n)$.

10.5 Ratio and Root Tests

Ratio Test: **Root Test:**

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \qquad \lim_{n\to\infty}\sqrt[n]{|a_n|} = L$$

For either test, $\sum_{n=k}^{\infty} a_n$ converges absolutely if $L < 1$, and diverges if $L > 1$.

The test is inconclusive if $L = 1$

10.6 Power Series

A power series with center c is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$.

Power series have a radius of convergence R such that they converge absolutely for $(c-R, c+R)$ and diverge for $(-\infty, c-R)\cup(c+R,\infty)$. The endpoints may converge or diverge. The ratio test is often useful to find the radius of convergence. $R = 0$ if the series converges only for $x = c$ and $R = \infty$ if the series converges $\forall\, x$. Power series can be differentiated or integrated by differentiating or integrating the general term.

8.4 Taylor Polynomials

$$T_n(x) = f(c) + \frac{f'(c)(x-c)^1}{1!} + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

The Maclaurin Polynomial is the special case where $c = 0$

$$\text{Error} \le \frac{|f^{(n+1)}(x)|\cdot|x-a|^{n+1}}{(n+1)!}$$

10.7 Taylor Series

Taylor series of $f(x)$ centered at $x = c$:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)(x-c)^n}{n!}$$

The Maclaurin series is the special case where $c = 0$

Function	Maclaurin Series		Converges
e^x	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$\forall\, x$
$\sin x$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$\forall\, x$
$\cos x$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$\forall\, x$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$\|x\| < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty}(-1)^n x^n$	$1 - x + x^2 - x^3 + \cdots$	$\|x\| < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \le 1$
$\arctan x$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 < x \le 1$
$(1+x)^a$	$\sum_{n=0}^{\infty}\binom{a}{n}x^n$	$1 + ax + \frac{a(a-1)x^2}{2!} + \cdots$	$\|x\| < 1$

11.1 Parametric Equations

Parametric curves are curves of the form $c(t) = (f(t), g(t))$.

Every curve C can be parameterized in infinitely many ways. Furthermore, $c(t)$ may traverse all or part of C more than once.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}, \quad x'(t) \ne 0$$

8.1 Arc Length and Surface Area

Arc Length:

$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

Surface Area of Revolution:

$$2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$

11.2 Arc Length and Speed

Arc Length:

$$\int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt$$

Surface Area of Revolution:

$$2\pi\int_a^b y(t)\sqrt{x'(t)+y'(t)^2}\,dt$$

Speed at time t:

$$\frac{ds}{dt}=\sqrt{x'(t)^2+y'(t)^2}$$

9.1 Solving Differential Equations

A differential equation has order n if $y^{(n)}$ is the highest-order derivative appearing in the equation.

A differential equation is linear if it can be written as:

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y$$

Separable first-order equation:

$$\frac{dy}{dx}=f(x)g(y)$$

Separation of Variables:

$$\frac{dy}{g(y)}=f(x)\,dx$$

$$\int\frac{dy}{g(y)}=\int f(x)\,dx$$

9.2 Models involving $y'=k(y-b)$

The solution of $y'=k(y-b)$ is $y=b+Ce^{kt}$ for constant C.

Equation $k>0$	Behavior
$y'=k(y-b)$	$\lim_{t\to\infty} y(t)=\infty$ if $C>0$, $\lim_{t\to\infty} y(t)=-\infty$ if $C<0$
$y'=-k(y-b)$	$\lim_{t\to\infty} y(t)=b$

11.3 Polar Coordinates

$x=r\cos\theta \quad y=r\sin\theta$

$r=\sqrt{x^2+y^2} \quad \tan\theta=\dfrac{y}{x}$

$(r,\theta)=(r,2\pi n), \quad n\in\mathbb{Z}$

$(-r,\theta)=(r,\theta+\pi)$

$(0,\theta_1)=(0,\theta_2)\,\forall\,\theta$

11.4 Area and Arc Length in Polar Coordinates

Area:

$$\frac{1}{2}\int_\alpha^\beta (f(\theta)^2-g(\theta)^2)\,d\theta, \quad f(x)\geq g(x)$$

Arc Length:

$$\int_\alpha^\beta \sqrt{f(\theta)^2+f'(\theta)^2}\,d\theta$$

12.7 Cylindrical and Spherical Coordinates

Cylindrical	Spherical	Rectangular
(r, θ, z)	(ρ, θ, ϕ)	(x, y, z)
$r = \sqrt{x^2 + y^2}$	$\rho = \sqrt{x^2 + y^2 + z^2}$	$x = r\cos\theta = \rho\cos\theta\sin\phi$
$\tan\theta = \frac{y}{x}$	$\tan\theta = \frac{y}{x}$	$y = r\sin\theta = \rho\sin\theta\sin\phi$
$z = z$	$\cos\phi = \frac{z}{\rho}$	$z = z = \rho\cos\phi$
$0 \le \theta < 2\pi,$	$0 \le \phi \le \pi$	