12.1 Vectors in the Plane

Position Vector:

$\mathbf{v} = p_2 - p_1 = \langle y_1 - x_1, \ldots, y_n - x_n \rangle$

Points $p_1 = (x_1, \ldots, x_n)$, $p_2 = (y_1, \ldots, y_n)$.

Magnitude:

$\|\mathbf{v}\| = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}, \quad \mathbf{v} = x_1, \ldots, x_n$

Equivalency:

Vectors which have the same magnitude and direction are *equivalent*.

Zero Vector:

$\mathbf{0} = \langle 0, \ldots, 0 \rangle, \quad \|\mathbf{0}\| = 0$

Vector Addition:

$\mathbf{x} + \mathbf{y} = \langle x_1 + y_1, \ldots, x_n + y_n \rangle$

Scalar Multiplication:

$k\mathbf{x} = \langle kx_1, \ldots, kx_n \rangle$.

The vector reverses direction for $k < 0$.

Parallelism:

$\mathbf{w} = c\mathbf{v} \implies \mathbf{w} \parallel \mathbf{v}$ for some scalar c.

Basis Vectors:

$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$

$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

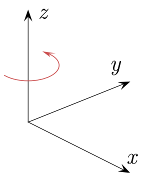
$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Triangle Inequality:

$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

12.2 Vectors in Three Dimensions

Right-hand Rule:



Line Parametrization:

$\mathbf{r}(t) = \mathbf{c} + t\mathbf{k}, \quad$ Constant vectors c, k of equal dimensions

Parametric Equations:

$x_j = c_j + tk_j$

Line through Two Points:

$\mathbf{r}(t) = (1-t)\mathbf{P}_1 + t\mathbf{P}_2, \quad$ Points $P_1 = p_{1_1}, \ldots, p_{1_n}, \quad P_2 = p_{2_1}, \ldots, p_{2_n}$

Constant Vectors $\mathbf{P}_1 = \langle P_1 \rangle, \quad \mathbf{P}_2 = \langle P_2 \rangle$

12.3 Dot Product and the Angle Between Two Vectors

Scalar Product:

$\mathbf{v} = v_1, \ldots, v_n, \quad \mathbf{w} = w_1, \ldots, w_n$

$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \ldots + v_n w_n = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta, \quad \theta = \mathbf{v} \measuredangle \mathbf{w}, \quad 0 \leq \theta \leq \pi$

$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

Angle Between Vectors:

$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}\right) = \arcsin\left(\frac{\mathbf{v} \times \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}\right)$

Range	$\mathbf{v}\cdot\mathbf{w}$	$\mathbf{v}\times\mathbf{w}$
$\theta = 0$	$\|\mathbf{v}\|\|\mathbf{w}\|$	0
$0 < \theta < \frac{\pi}{2}$	$+$	$+$
$\theta = \frac{\pi}{2}$	0	$\|\mathbf{v}\|\|\mathbf{w}\|$
$\frac{\pi}{2} < \theta < \pi$	$-$	$+$
$\theta = \pi$	$-\|\mathbf{v}\|\|\mathbf{w}\|$	0
$\pi < \theta < \frac{3\pi}{2}$	$-$	$-$
$\theta = \frac{3\pi}{2}$	0	$-\|\mathbf{v}\|\|\mathbf{w}\|$
$\frac{3\pi}{2} < \theta < 2\pi$	$+$	$-$

Properties of Scalar Products:

Commutativity	$\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$
Associativity	$(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$
Distributivity	$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
Scalar Factorization	$\lambda\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\cdot\lambda\mathbf{w} = \lambda(\mathbf{v}\cdot\mathbf{w})$

Projection:

$$\mathbf{w}_{\mathbf{v}} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{v}\|}\mathbf{v} = (\mathbf{w}\cdot\mathbf{e}_{\mathbf{v}})\mathbf{e}_{\mathbf{v}},$$

Projection $\mathbf{w}_{\mathbf{v}}$ of $\mathbf{w}$ along $\mathbf{v}$ Unit vector $\mathbf{e}_{\mathbf{v}}$ in the direction of $\mathbf{v}$

$$\mathbf{w} = \mathbf{w}_{\parallel} + \mathbf{w}_{\perp}, \quad \mathbf{v} \neq 0, \quad (\mathbf{w}_{\parallel} = \mathbf{w}_{\mathbf{v}}) \parallel \mathbf{v}, \quad \mathbf{w}_{\perp} \perp \mathbf{v}$$

Unit Vectors:

$$\mathbf{e}_{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \quad \|\mathbf{e}_{\mathbf{v}}\| = 1 \quad \mathbf{v} \neq 0.$$

Unit Vector in Two Dimensions:

$$\mathbf{e}_{\mathbf{v}} = \langle \cos\theta, \sin\theta\rangle, \quad \theta = \mathbf{u}\measuredangle\mathbf{i}$$

Components:

$\mathbf{w}\cdot\mathbf{e}_v = \|\mathbf{w}\|\cos\theta$ *Component* $\mathbf{w}\cdot\mathbf{e}_{\mathbf{v}}$ of $\mathbf{w}$ along $\mathbf{v}$:

12.4 The Cross Product

2×2 **Determinant:**

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

3×3 **Determinant:**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Vector Product:

$\mathbf{v} = \langle a_1, b_1, c_1\rangle$, $\mathbf{w} = \langle a_2, b_2, c_2\rangle$

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Properties of Vector Products:

Anti-Commutativity	$\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}$
Non-Associativity	$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} \neq \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$
Distributivity over Addition	$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
Scalar Factorization	$\lambda\mathbf{v}\times\mathbf{w} = \mathbf{v}\times\lambda\mathbf{w} = \lambda(\mathbf{v}\times\mathbf{w})$

The cross product is a three dimensional operation:

$$(\mathbf{v}\times\mathbf{w}) \in \mathbb{R}^3, \quad \mathbf{v} \in \mathbb{R}^3, \quad \mathbf{w} \in \mathbb{R}^3$$

Mutual Orthogonality:

$(\mathbf{v}\times\mathbf{w}) \perp$ both $\mathbf{v}$ and $\mathbf{w}$

Vector Product Magnitude:

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

$\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2$

Scalar Triple Product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Spanned Area:

Parallelogram area spanned by $\mathbf{v}$ and $\mathbf{w}$: $\|\mathbf{v} \times \mathbf{w}\|$

Parallelepiped volume spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

12.5 Planes in Three-Space

Equation of a Plane:

Plane through point $P = (p_1, \ldots, p_n)$ Constant vector $\mathbf{P} = \langle P \rangle$,

Constant vector $\mathbf{n} = \langle k_1, \ldots, k_n \rangle$ normal to plane,

$\mathbf{x} = \langle x_1, \ldots, x_n \rangle$:

Vector Form $\mathbf{n}\,(\mathbf{x} - \mathbf{P}) = 0$

Scalar Form $k_1(x_1 - p_1) + \ldots + k_n(x_n - p_n) = 0$

Planes are parallel if they have the same $\mathbf{n}$.

Traces:

The projection onto a coordinate plane is called a *trace*.

To find a trace, set variables that are not within the coordinate plane to 0.

12.7 Cylindrical and Spherical Coordinates

$x, y, z \to r, \theta, z$	$x, y, z \to \rho, \theta, \phi$	$r, \theta, z \to x, y, z$	$\rho, \theta, \phi \to x, y, z$
$r = \sqrt{x^2 + y^2}$	$\rho = \sqrt{x^2 + y^2 + z^2}$	$x = r\cos\theta$	$x = \rho\cos\theta\sin\phi$
$\theta = \arctan\left(\frac{y}{x}\right)$	$\theta = \arctan\left(\frac{y}{x}\right)$	$y = r\sin\theta$	$y = \rho\sin\theta\sin\phi$
$z = z$	$\phi = \arccos(z)$	$z = z$	$z = \rho\cos\phi$
$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$		
	$0 \le \phi \le \pi$		

Sphere:

$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ Radius r, Center (a, b, c)

Cylinder:

$(x - a)^2 + (y - b)^2 = r^2$ Radius r, Vertical axis through $(a, b, 0)$

Integration in Polar Coordinates:

$$\int\int_{\mathcal{D}} f(x, y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

Integration in Cylindrical Coordinates:

$$\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

Integration in Spherical Coordinates:

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Cylinder Parameterization, radius R:

$\mathbf{r}(\theta, z) = (R\cos\theta, R\sin\theta, z)$

Outward Normal:

$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = R\langle \cos\theta, \sin\theta, 0 \rangle$

$dS = \|\mathbf{n}\|\, d\theta\, dz = R\, d\theta\, dz$

Sphere Parameterization, radius R:

$\mathbf{r}(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$

Unit radial vector:

$\mathbf{e}_r = \langle \cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$

Outward Normal:

$\mathbf{n} = \mathbf{T}_\phi \times \mathbf{T}_\theta = (R^2 \sin\phi)\mathbf{e}_r$

$dS = \|\mathbf{n}\|\, d\phi\, d\theta = R^2 \sin\phi\, d\phi\, d\theta$

Parameterization for $z = f(x, y)$:

$\mathbf{r}(x, y, f(x, y))$

$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \langle -f_x, -f_y, 1\rangle$

$dS = \|\mathbf{n}\|\, dx\, dy = \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy$

13.1 Vector Functions

Vector Parameterization of a Path:

$\mathbf{r}(t) = \langle x_1(t), \ldots, x_n(t)\rangle$

A *curve* is set of all points in the domain of a path.

Curves are not unique. Every curve can be parametrized in infinitely many ways.

13.2 Calculus of Vector Functions

Limits:

$\lim_{t\to c} \mathbf{F}(t) = \left\langle \lim_{t\to c} F_1(t), \ldots, \lim_{t\to c} F_n(t)\right\rangle$ Performed componentwise.

Differentiation:

$\frac{\partial}{\partial t}\mathbf{F}(t) = \left\langle \frac{\partial}{\partial t}F_1(t), \ldots, \frac{\partial}{\partial t}F_n(t)\right\rangle$ Performed componentwise.

Integration:

$\int \mathbf{F}(t)\, dt = \left\langle \int F_1(t)\, dt, \ldots, \int F_n(t)\, dt\right\rangle$ Performed componentwise.

Sum Rule:

$\frac{\partial}{\partial t}(\mathbf{r}_1(t) + \ldots + \mathbf{r}_n(t)) = \frac{\partial}{\partial t}\mathbf{r}_1(t) + \ldots + \frac{\partial}{\partial t}\mathbf{r}_n(t)$

Constant Multiple Rule:

$\frac{\partial}{\partial t}(c\mathbf{r}(t)) = c \cdot \frac{\partial}{\partial t}\mathbf{r}(t)$

Chain Rule:

$\frac{\partial}{\partial t}\mathbf{r}(g(t)) = \frac{\partial}{\partial t}g(t) \cdot \frac{\partial}{\partial t}\mathbf{r}(g(t))$

Product Rule:

Scalar times Vector	$\frac{\partial}{\partial t}(f(t)\mathbf{r}(t)) = f(t) \cdot \frac{\partial}{\partial t}\mathbf{r}(t) + \frac{\partial}{\partial t}f(t) \cdot \mathbf{r}(t)$
Scalar Product	$\frac{\partial}{\partial t}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1(t) \cdot \frac{\partial}{\partial t}\mathbf{r}_2(t) + \frac{\partial}{\partial t}\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)$
Vector Product	$\frac{\partial}{\partial t}(\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{r}_1(t) \times \frac{\partial}{\partial t}\mathbf{r}_2(t) + \frac{\partial}{\partial t}\mathbf{r}_1(t) \times \mathbf{r}_2(t)$

Velocity Vector:

$\mathbf{v}(t) = \frac{\partial}{\partial t}\mathbf{r}(t),$ Also called the *tangent vector.*

Speed:

$v(t) = \|\mathbf{v}(t)\|$

Parametrization of the Tangent Line:

$\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{v}(t_0),$ At time $t = t_0$, Velocity vector $\mathbf{v}(t)$

$\frac{\partial}{\partial t}\mathbf{r}_1(t) = \frac{\partial}{\partial t}\mathbf{r}_2(t) \implies \mathbf{r}_1(t) = \mathbf{r}_2(t) + \mathbf{c},$ Constant vector $\mathbf{c}$

The Fundamental Theorem of Calculus for

Vector Functions:

If $\mathbf{r}(t)$ is continuous and $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$\int_a^b \mathbf{r}(t)\, dt = \mathbf{R}(b) - \mathbf{R}(a)$

13.3 Arc Length and Speed

Arc Length:

$\int_{t_1}^{t_2} \left\| \frac{\partial}{\partial t}\mathbf{r}(t)\right\| dt$

Arc Length Parametrization:

$v(t) = 1\ \forall\ t \implies \mathbf{r}(t)$ is an *arc length parametrization*

$\mathbf{v}(t) \neq \mathbf{0}\ \forall\ t \implies \mathbf{r}_1(s) = \mathbf{r}(g(s)),$

Where $t = g(s)$ is the inverse of the arc length function.

13.4 Curvature

Tangent Vector	**Normal Vector**	**Binormal Vector**
$\mathbf{N} \times \mathbf{B}$ $\mathbf{T}(t) = \mathbf{v}(t)$	$\mathbf{B} \times \mathbf{T}$ $\mathbf{N}(t) = \mathbf{a}(t) = \nabla \mathbf{r}(t)$	
Unit Tangent Vector	**Unit Normal Vector**	$\hat{\mathbf{T}} \times \hat{\mathbf{N}}$
$\hat{\mathbf{N}} \times \hat{\mathbf{B}}$ $\hat{\mathbf{T}}(t) = \mathbf{e_v}(t)$	$\hat{\mathbf{B}} \times \hat{\mathbf{T}}$ $\hat{\mathbf{N}}(t) = \mathbf{e_a}(t) = \dfrac{\nabla \mathbf{r}(t)}{\lVert \nabla \mathbf{r}(t) \rVert}$	
Normal Plane	**Rectifying Plane**	**Osculating Plane**
$\mathbf{T}\,(\mathbf{x} - \mathbf{P}) = 0$	$\mathbf{N}\,(\mathbf{x} - \mathbf{P}) = 0$	$\mathbf{B}\,(\mathbf{x} - \mathbf{P}) = 0$

Velocity Vector $\mathbf{v}$, Unit Velocity Vector $\mathbf{e_v}$,

Acceleration Vector $\mathbf{a}$, Unit Acceleration Vector $\mathbf{e_a}$,

Constant vector $\mathbf{P} = p_1, \ldots, p_n \quad \mathbf{x} = \langle x_1, \ldots, x_n \rangle$

Curvature for Vector Functions

$$\kappa(t) = \left\lVert \frac{\partial \mathbf{T}}{\partial t} \right\rVert = \frac{\lVert \mathbf{v}(t) \times \mathbf{a}(t) \rVert}{\lVert \mathbf{v}(t) \rVert^3}$$

Curvature for Scalar Functions:

$$\kappa(x) = \frac{\lVert f''(x) \rVert}{(1 + f'(x)^2)^{3/2}}$$

Regularity:

Parametrization $\mathbf{r}(t) \neq \mathbf{0} \; \forall \, t \implies \mathbf{r}$ is *regular.*

$\mathbf{T}'(t) = \kappa(t) v(t) \mathbf{N}(t)$

Osculating Circle:

The circle with radius $\dfrac{1}{\kappa_P}$ and center in the $\mathbf{N}$ direction.

13.5 Motion in Three-Space

Acceleration Vector:

$\mathbf{a}(t) = \mathbf{r}''(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t)$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = v'(t) = \left(\frac{\mathbf{a} \cdot \mathbf{T}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \qquad a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \kappa(t) v(t)^2 = a_{\mathbf{N}} \mathbf{N} = \mathbf{a} - a_{\mathbf{T}} \mathbf{T}$$

14.1 Functions of Two or More Variables

Domain:

The domain of a function f of n variables is the set of points $P \in \mathbb{R}^n$ for which $f(P)$ is defined.

Range:

The range of f is the set of values taken by f.

Graph:

The graph of a continuous real-valued function $f(\mathbf{x})$ is the surface in $\mathbb{R}^{n+1}$ consisting of the points $(\mathbf{x}, f(\mathbf{x}))$ for $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ in the domain of f.

14.2 Limits and Continuity in Several Variables

Open Disk of radius r centered at $P = (c_1, \ldots, c_n)$:

$D(P, r) = \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - c_1)^2 + \ldots + (x_n - c_n)^2 < r^2\}$

Punctured Disk:

$D^*(P, r) = D(P, r) - P.$

Interior:

A point P is an interior point of a domain $\mathcal{D}$ if $\mathcal{D}$ contains some open disk $D(P, r)$ centered at P. The interior of $\mathcal{D}$ is the set of all interior points.

Boundary:

A point P is a boundary point of a domain $\mathcal{D}$ if every open disk $D(P, r)$ contains points both in and not in $\mathcal{D}$. The boundary of $\mathcal{D}$ is the set of all boundary points.

14.3 Partial Derivatives

$$f_{x_j}(\mathbf{x}) = \frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_j + h, \ldots, x_n) - f(\mathbf{x})}{h}$$

Partial Derivative Approximation:

$f(x_1,\dots,x_j+\Delta x_j,\dots,x_n)-f(\mathbf{x})\approx f_{x_j}(\mathbf{x})\Delta x_j$

Second-order Partial Derivatives:

$f_{xx}=\dfrac{\partial^2}{\partial x^2}f \quad f_{xy}=\dfrac{\partial^2}{\partial y\partial x}f \quad f_{yy}=\dfrac{\partial^2}{\partial y^2}$

Clairaut's Theorem:

Mixed partial derivatives can be computed in any order.

14.4 Differentiability, Linear Approximation, and Tangent Planes

Linearization:

$L(\mathbf{x})=f(P)+\nabla f_P\cdot(\mathbf{x}-\mathbf{P}),$

Point $P=(p_1,\dots,p_n)$, Constant Vector $\mathbf{P}=\langle P\rangle$, $\mathbf{x}=\langle x_1,\dots,x_n\rangle$

Tangent Plane:

The linearization of function f at point P is the plane tangent to f at P

Linear Approximation:

$f(P)\approx L(P)$ A function at point P is approximate to its linearization at P.

$f(x_1,\dots,x_j+c,\dots,x_n)\approx f+f_{x_1}+\dots+cf_{x_j}+\dots+f_{x_n}$

$\Delta f\approx\nabla f\cdot\partial x_1\dots\partial x_n$

Multivariable Differentiability:

$f:\mathbb{R}^m\to\mathbb{R}^n$ is differentiable if there exists a Jacobian $J:\mathbb{R}^m\to\mathbb{R}^n$ such that:

$$\lim_{h\to 0}\frac{f(P+h)-f(P)-J(h)}{\|h\|}=0$$

14.5 The Gradient and Directional Derivatives

The Gradient:

$$\nabla f=\left\langle\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right\rangle$$

∇f_P is directed to the maximum rate of increase, magnitude equal to this rate.

$-\nabla f_P$ is directed to the maximum rate of decrease, magnitude equal to this rate.

Chain Rule for Paths:

$$\frac{\partial}{\partial t}f(\mathbf{c}(t))=\nabla f_{\mathbf{c}(t)}\cdot\mathbf{c}'(t)$$

Directional Derivative:

$$D_{\mathbf{v}}f(P)=\lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{v})-f(\mathbf{x})}{h}=\nabla f_P\cdot\mathbf{e_v}=\|\nabla f_P\|\cos\theta$$

$\theta=\nabla f_P\measuredangle\mathbf{v}$, $f(\mathbf{x})=f(x_1,\dots,x_n)$, Unit vector $\mathbf{e_v}$ of $\mathbf{v}$

14.6 The Chain Rule

General Case:

$$f_{t_k}=\nabla f\cdot\left\langle\frac{\partial x_1}{\partial t_k},\dots,\frac{\partial x_n}{\partial t_k}\right\rangle,\quad f(x_1(t_1,\dots,t_m),\dots,x_n(t_1,\dots,t_m))$$

Partial Derivatives:

$\dfrac{\partial z}{\partial x_j}=-\dfrac{f_{x_j}}{f_z}$ when z is defined implicitly by $f(\mathbf{x},z(\mathbf{x}))=0$

14.7 Optimization in Several Variables

Critical Point:

P is a critical point of $f(\mathbf{x})$ if $f_{x_j}(P)=0$ or $\nexists$, $\forall\, x_j$

Local Extrema:

The local minima and local maxima are in the set of critical points.

The Hessian Matrix:

$$\text{Hessian Matrix } \mathbf{H}(f)(\mathbf{x})=\mathbf{J}(\nabla f)(\mathbf{x})=\begin{bmatrix}\frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial x_n\partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2}\end{bmatrix}$$

Discriminant:

Discriminant D of $f(\mathbf{x})$ at point P:

$D(P)=\mathbf{H}_f(P)$, Hessian Determinant $\mathbf{H}_f$

Second Partial Derivative Test:

For critical point of f:

$\mathbf{H}$ is positive definite	Local Minimum
$\mathbf{H}$ is negative definite	Local Maximum
$D < 0$	Saddle Point
$D = 0$	Inconclusive

14.8 Lagrange Multipliers: Optimizing with a Constraint

The extrema of $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ subject to constraints $\mathbf{g}(\mathbf{x}) = 0$:

$$\nabla f = \sum_{j=1}^{n} \lambda_j \nabla g_j$$

This equation also applies to partial derivatives.

If the constraint is bounded, then global extrema subject to the constraint exist.

15.1 Integration in Two Variables

Riemann Sum in Multiple Variables:

$$R_S = \sum_{j=1}^{n} f(t_j)(x_{j+1} - x_j),$$

Multiple Integral:

$$\underbrace{\int \ldots \int_{\mathcal{D}}}_{dick} f(\mathbf{x})\, dA = \lim_{m,n\to\infty} R_S, \quad dA = dx_1 \ldots dx_n$$

Closed, bounded domain $\mathcal{D}$, whose boundary is:

A simple curve (one-to-one) on each axis, and is either

smooth or has a finite number of corners.

f is differentiable if this limit exists.

Continuity of f on $\mathcal{D} \implies$ integrability.

Integration can be done in any order.

Integral of a Constant function:

$$\int \ldots \int_{\mathcal{D}} C\, dA = C \cdot \text{Area}(\mathcal{D}), \quad \text{Constant } C$$

Non-simple Domains:

If the domain of the variable being integrated is non-simple,

first integrate over a variable with a simple domain.

Riemann Domain Additivity:

$$\int \ldots \int_{\mathcal{D}} f\, dA \approx \sum_{j=1}^{N} f(P_j) \cdot \text{Area}(\mathcal{D}_j),$$

Union of N small, nonoverlapping subdomains $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_N$

Domain Additivity:

$$\int \ldots \int_{\mathcal{D}} f\, dA = \sum_{j=1}^{N} \int \ldots \int_{\mathcal{D}_j} f\, dA,$$

Union of N nonoverlapping subdomains $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_N$

Average Value:

$$\overline{f} = \frac{1}{\text{Area}(\mathcal{D})} \cdot \int \ldots \int_{\mathcal{D}} f\, dA$$

Mean Value Theorem for Integrals:

If f is differentiable $\forall\, P \in \mathcal{D}\ \exists\, P : f(P) = \overline{f} \to \mathcal{D}$

Integrals of Greater Functions:

$$f \geq g \in \mathcal{D} \implies \int \ldots \int_{\mathcal{D}} f\, dA \geq \int \ldots \int_{\mathcal{D}} g\, dA$$

A greater function implies a greater integral.

Integral Bounds:

$$m \cdot \text{Area}(\mathcal{D}) \leq \int \ldots \int_{\mathcal{D}} f\, dA \leq M \cdot \text{Area}(\mathcal{D}),$$

$m = \min(f) \in \mathcal{D} \quad M = \max(f) \in \mathcal{D}$

The integral is between the maximum of f on $\mathcal{D}$ times the unsigned area of $\mathcal{D}$,

and the minimum of f on $\mathcal{D}$ times the unsigned area of $\mathcal{D}$

15.2 Triple Integrals

15.3 Integration in Polar, Cylindrical, and Spherical Coordinates

15.6 Change of Variables

The Jacobian Matrix:

Let $\mathbf{F}(F_1(\mathbf{x}), \ldots, F_m(\mathbf{x})) : \mathbb{R}^n \to \mathbb{R}^m$

be a differentiable and one-to-one on the interior of $\mathbb{R}^n$

Jacobian matrix $\mathbf{J}(\mathbf{F})(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$

Where $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$

Change of Variables Formula:

Let $f(\mathbf{x}_1)$ be a continuous function.

Let $\mathbf{F}(F_1(\mathbf{x}_2), \ldots, F_m(\mathbf{x}_2)) : \mathbb{R}^n \to \mathbb{R}^m$

be a differentiable and one-to-one on the interior of $\mathbb{R}^n$

$$\int \cdots \int_{\mathbb{R}^m} f(\mathbf{x}_1)\, d\mathbf{x}_1 = \int \cdots \int_{\mathbb{R}^n} \mathbf{F} \cdot |\mathbf{J}_\mathbf{F}|\; d\mathbf{x}_2$$

$\mathbf{x}_1 = x_{1_1}, \ldots, x_{1_m}, \quad \mathbf{x}_2 = x_{2_1}, \ldots, x_{2_n}, \quad d\mathbf{x} = dx_1 dx_2 \ldots$

Jacobian determinant $\mathbf{J}_\mathbf{F}$,

$\mathbb{R}^m = f(\mathbb{R}^n)$ is the image of the original region $\mathbb{R}^n$

Jacobian Identities:

$\mathbf{J}_\mathbf{F} = \mathbf{J}^{-1}_{\mathbf{F}^{-1}}$

16.1 Vector Fields

Vector Field:

A vector field assigns a vector to each point in a domain.

A vector field in $\mathbb{R}^n$ is represented by a vector of n smooth functions:

$\mathbf{F} = \langle F_1, \ldots, F_n \rangle$

Radial Unit Vector Field:

$\mathbf{e}_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \nabla r$

Inverse-Square Vector Field:

$\frac{\mathbf{e}_r}{r^2} = \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle = \nabla(-r^{-1})$

16.2 Line Integrals

Scalar Line Integral	**Vector Line Integral**
$\int_\mathcal{C} f\, ds = \int_a^b f(\mathbf{c}(t)) \lVert \mathbf{c}'(t) \rVert\, dt$	$\int_\mathcal{C} \mathbf{F} \cdot d\mathbf{s} = \int_\mathcal{C} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$

For Regular Vector Parametrization $\mathbf{c}(t)$, $\quad a \leq t \leq b$, $\quad$ Oriented Curve $\mathcal{C}$

Line element $ds = \lVert \mathbf{c}'(t) \rVert\, dt \qquad$ Vector Differential $d\mathbf{s} = \langle dx_1, \ldots, dx_n \rangle$

Oriented Curves:

An oriented curve is a curve with a defined positive direction.

$$\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -\int_\mathcal{C} \mathbf{F} \cdot d\mathbf{s}$$

16.3 Conservative Vector Fields

Conservative Vector Fields:

$\mathbf{F}$ on domain $\mathcal{D}$ is conservative $\iff \exists\, V : \nabla V = \mathbf{F}$ on $\mathcal{D}$.

V is a potential function for $\mathbf{F}$

Fundamental Theorem for Conservative Vector Fields:

$$\int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s} = V(Q) - V(P) \implies \oint_\mathbf{c} \mathbf{F} \cdot d\mathbf{s} = 0$$

$\forall \quad$ Points $P, Q, \quad$ Paths $\mathbf{c} \in \mathcal{D}$ from P to Q.

$V_1 = cV_2$ for any two potential functions for a conservative vector field.

Conservative vector fields satisfy the following condition:

$$\frac{\partial F_j}{\partial x_{j+1}} = \frac{\partial F_{j+1}}{\partial x_j}, \quad \mathbf{F} = \langle F_1(\mathbf{x}), \ldots, F_m(\mathbf{x}) \rangle$$

Path Independence:

$\mathbf{F}$ on domain $\mathcal{D}$ is path-independent $\iff \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$

$\forall \quad$ Points $P, Q, \quad$ Paths $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{D}$ from P to Q.

Path Independent $\iff$ Conservative

16.4 Parameterized Surfaces and Surface Integrals

Parametric Hypersurface:

$$\mathbf{r}(\mathbf{x}) = \langle y_1(\mathbf{x}), \ldots, y_n(\mathbf{x}) \rangle$$

Tangent Vector	Normal Vector	Unit Normal Vector
$\mathbf{T}_{x_j} = \mathbf{r}_{x_j}$	$\mathbf{n} = \mathbf{T}_{x_1} \times \ldots \times \mathbf{T}_{x_n}$	$\frac{\mathbf{n}}{\\|\mathbf{n}\\|}$

Regularity:

A parametric surface is regular at point P if $\mathbf{n}(P) \neq 0$

Surface Area Approximation:

For a surface $\mathcal{S} = \mathbf{r}(\mathcal{D})$ with normal vector $\mathbf{n}$:

$$\text{Area}(\mathcal{S}) \approx \|\mathbf{n}(P)\| \cdot \text{Area}(\mathcal{D}), \quad P \in \mathcal{D}.$$

Surface Area by Riemann Sum:

$$\text{Area}(\mathcal{S}) = \lim_{m_1, \ldots, m_n \to \infty} \sum_{i=1}^{m_1} \ldots \sum_{j=1}^{m_n} \mathbf{n}(x_{1i}, \ldots, x_{nj}) \Delta x_1 \ldots \Delta x_n,$$

Surface $\mathcal{S} = \mathbf{r}(\mathcal{R}), \quad$ Hyperrectangular Domain $\mathcal{R}$

Surface Area:

$$\text{Area}(\mathcal{S}) = \underbrace{\int \ldots \int_{\mathcal{D}}}_{n} \|\mathbf{n}(\mathbf{x})\| \, dx_1 \ldots dx_n = \int_{\mathcal{S}} 1 \, dS,$$

Surface Integral over a Function:

$$\int_{\mathcal{S}} f(\mathbf{y}) \, dS = \underbrace{\int \ldots \int_{\mathcal{D}}}_{n} f(\mathbf{r}(\mathbf{x})) \|\mathbf{n}(\mathbf{x})\| \, dx_1 \ldots dx_n$$

16.5 Surface Integrals of Vector Fields

Orientation:

A surface $\mathcal{S}$ is *oriented* if a continuously varying unit normal vector $\mathbf{e_n}(P)$ is specified at each point on $\mathcal{S}$. This distinguishes an outward direction on the surface.

The integral of a vector field $\mathbf{F}$ over an oriented surface $\mathcal{S}$ is defined as the integral of the normal component $\mathbf{F} \cdot \mathbf{e_n}$ over $\mathcal{S}$.

Vector surface integral:

$$\int \ldots \int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \underbrace{\int \ldots \int_{\mathcal{D}}}_{dick} \mathbf{F}(G(u,v)) \cdot \mathbf{n}(u,v) \, du \, dv$$

Parametrization $G(u,v)$ of $\mathcal{S}$ such that $\mathbf{n}(u,v) = \mathbf{T}_u \times \mathbf{T}_v$ points in the direction of the unit normal vector specified by the orientation. This is also called the *flux* of $\mathbf{F}$ through G.

17.1 Green's Theorem

Line Integral of a Vector Field:

$$\int_{\mathcal{C}} F_1 \, dx_1 + \ldots + F_n \, dx_n$$

Green's Theorem:

$$\oint_{\partial \mathcal{D}} F_1 \, dx_1 + \ldots + F_n \, dx_n = \underbrace{\int \ldots \int_{\mathcal{D}}}_{dick} \text{curl}(\mathbf{F}) \, dA$$

Area of a region enclosed by a Curve:

$$\text{Area}(\mathcal{D}) = \frac{1}{2} \oint_{\mathcal{C}} = x \, dy - y \, dx$$

Closed Line Integral Approximation:

$$\oint_{\mathcal{C}} F_1\,dx_1 + \ldots + F_n\,dx_n \approx \text{curl}(\mathbf{F})(P)\cdot\text{Area}(\mathcal{D}),$$

Domain $\mathcal{D}$ bounded by $\mathcal{C}$, point $P \in \mathcal{D}$

17.2 Stokes' Theorem

Surface Boundary:

Denoted $\partial\mathcal{S}$ $\quad\mathcal{S}$ is *closed* if $\partial\mathcal{S}$ is empty.

Circulation per unit Area:

$$\text{curl}(\mathbf{F}) = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Del Operator:

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$$

Stokes' Theorem:

$$\oint_{\partial\mathcal{S}} \mathbf{F}\cdot d\mathbf{s} = \int\ldots\int_{\mathcal{S}} \text{curl}(\mathbf{F})\cdot d\mathbf{S}$$

$\mathbf{F} = \nabla V \implies \text{curl}(\mathbf{F}) = \mathbf{0}$

Surface Independence:

$F = \text{curl}(\mathbf{A}) \implies$ flux of $\mathbf{F}$ through $\mathcal{S}$ depends on $d\mathcal{S}$ and not on $\mathcal{S}$.

$$\int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{A}\cdot d\mathbf{s}$$

If $\mathcal{S}$ is closed and $\mathbf{F} = \text{curl}(\mathbf{A}) \implies \int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = 0$

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} \approx (\text{curl}(\mathbf{F})(P)\cdot\mathbf{e_n})\,\text{Area}(\mathcal{D})$$

Point P, unit normal vector $\mathbf{e_n}$, simple closed curve $\mathcal{C}$ around P in the plane through P, enclosed region $\mathcal{D}$